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# Vector bundle representations of groups in quantum physics

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**Abstract.** The theory of vector bundle representations of Lie groups $G$ is developed. Induced and locally operating linear representations of $G$ are shown to be generated by vector bundle representations, and gauge equivalence of locally operating linear representations is shown to correspond to equivalence of vector bundle representations. It is pointed out that every vector bundle representation is equivalent to an induced one. Some applications to the quantum systems associated with monopole and instanton gauge field configurations are also discussed.

## 1. Introduction

In a series of papers Hoogland (1976, 1977, 1978) pointed out that the representations of groups of transformations of a space–time which are relevant for quantum mechanics are those christened by him as locally operating representations. Moreover, he has shown that the concept of equivalence of representations must be modified to take into account the local character of these representations, which leads to the new concept of local (gauge) equivalence.

The aim of this paper is to give an intrinsic geometric formulation of such representations and their gauge equivalence that generalises the theory of induced representations. In this formulation the (linear) locally operating representations are described in terms of vector bundle representations. By developing the theory of vector bundle representations we generalise Hoogland's concept of locally operating (linear) representations when the corresponding vector bundle is not trivial. Vector bundle representations have already been used in quantum theory by Sen (1975, 1978) in a different context. Our main results (§§ 2, 3) ensure that any vector bundle representation is equivalent to an induced one, and that the equivalence of vector bundle representations is related to the equivalence of the corresponding induced ones. Some new applications to the monopole, instanton and the physics of two identical particles are given in § 4. The framework of vector bundle representations that we develop in this paper allows us to deal in a similar way with locally operating projective representations which will be carried out in a subsequent paper.

 

## 2. Locally operating representations of transformation groups

Let $G$ be a connected Lie group acting transitively on the left on a manifold $M$. Following Hoogland (1976, 1977, 1978), we will call (linear) locally operating representations of $G$ the (linear) representations of $G$ which operate locally on a space $\mathscr{F}$ *of* $\mathbb{C}^m$-valued functions defined on $M$, according to the following law:

$$\mathscr{U}(g)F(gx)=A(g,x)F(x) \tag{2.1}$$

where $A$ is a matrix valued function $A: G\times M\to \mathrm{GL}(m,\mathbb{C})$ verifying the composition law

$$A(g_1,g_2x)A(g_2,x)=A(g_1g_2,x). \tag{2.2}$$

The matrix $A$ is called the gauge matrix.

When $M$ is a physical space–time these representations are relevant because they preserve locality in $M$. For instance, covariant representations of Poincaré-like groups are but particular examples of such representations, where $A$ does not depend on $x$.

The natural concept of equivalence of locally operating representations is the so-called gauge equivalence: two locally operating representations of $G$, $\mathscr{U}$ and $\mathscr{U}'$, are said to be gauge equivalent if there is an invertible transformation $T$ acting locally in $\mathscr{F}$ (i.e. $(TF)(x)=S(x)F(x)$, with $S$ being a function $S:M\to \mathrm{GL}(m,\mathbb{C})$), such that $\forall g\in G$,

$$\mathscr{U}'(g)=T\mathscr{U}(g)T^{-1}. \tag{2.3}$$

The corresponding gauge matrices will be related by

$$A'(g,x)=S(gx)A(g,x)S^{-1}(x). \tag{2.4}$$

Let $x_0$ be an arbitrary but fixed point of $M$ and $G_{x_0}$ the corresponding isotopy group. Let us suppose that the principal fibre bundle $G(M,G_{x_0})$ admits a global section $s$. Then the representations of $G$ induced from those of $G_{x_0}$, when expressed in terms of $s$ (see e.g. Simms 1968), give rise to locally operating representations of $G$ as was pointed out by Cariñena *et al* (1982); in that paper it was also claimed that any locally operating representation of $G$ is gauge equivalent to some of these induced representations. The proof of this claim is given by the following theorem.

*Theorem 1.* Let $\mathscr{U}$ be the locally operating representation of $G$ in $\mathscr{F}$ given by (2.1). Then the map $\Sigma: G_{x_0}\to \mathrm{GL}(m,\mathbb{C})$ given by $\Sigma(\gamma)=A(\gamma,x_0)$ defines a linear representation of $G_{x_0}$ and the representation of $G$ induced from $\Sigma$, when expressed in terms of the global section $s$, is a locally operating representation of $G$ gauge-equivalent to $\mathscr{U}$.

*Proof.* Let $s$ be a normalised global section of the principal fibre bundle $G(M,G_{x_0})$, i.e. $s(x)x_0=x$, $\forall x\in M$ and $s(x_0)$ is the identity of $G$. Let us now consider the action of $G$ on the functions of $M$ in $\mathbb{C}^m$ which is associated to the representation of $G$ induced from $\Sigma$ by making use of the trivialisation of $G(M,G_{x_0})$ provided by the section $s$. Such an action defines a locally operating representation whose gauge matrix is given by

$$A'(g,x)=\Sigma(s^{-1}(gx)gs(x))=A(s^{-1}(gx)gs(x),x_0). \tag{2.5}$$

If we take into account the composition law of such gauge matrices we can write

$$A'(g,x)=A(s^{-1}(gx),gx)A(g,x)A(s(x),x_0)$$

and therefore the functions $S(x)=A(s^{-1}(x),x)=A^{-1}(s(x),x_0)$ allow us to define a gauge equivalence between $\mathscr{U}$ and the representation of $G$ induced from $\Sigma$.

Notice that the section $s$ determines a map $\gamma: G \to G_{x_0}$, by the factorisation $g = s(gx_0)\gamma(g)$. When the gauge matrix $A$ satisfies $A(g, x_0) = A(\gamma(g), x_0)$, it is said to be $x_0$-centred (with respect to the section $s$). From the proof of theorem 1 it follows that a locally operating representation of $G$ with an $x_0$-centred gauge matrix (with respect to the section $s$) coincides with the expression of the representation of $G$ induced from the representation of $G_{x_0}$ given by $\Sigma(h)=A(h, x_0)$ if the trivialisation associated to $s$ is used, because $s(x)=I$.

However, there are locally operating realisations of groups which are non-centred with respect to any section, and therefore the concept of locally operating representation is more general than that of induced representation. On the other hand, induced representations of groups have been formulated in an intrinsic framework by means of the natural action of $G$ on the corresponding associated fibre bundle, with no use of sections (Bott 1965, Simms 1968). In the general case, a (continuous) global section does not exist (see e.g. Boya *et al* 1974, and references therein). Hence we would look for a similar intrinsic formulation of locally operating representations which does not make use of global sections and where topological features which were deliberately forgotten in Hoogland's papers are taken into account. Such topological aspects have been shown to be physically relevant in field theory (monopoles, instantons etc). The appropriate geometrical setting for such a formulation is given in § 3.

## 3. Vector bundle representations of $G$

Let $E(X, \pi_E, \mathbb{C}^m)$ denote a differentiable vector bundle with total space $E$, base space $X$, projection $\pi_E$ and typical fibre $\mathbb{C}^m$. We recall that a morphism of the vector bundle $E(X, \pi_E, \mathbb{C}^m)$ in the vector bundle $E'(X', \pi_{E'}, \mathbb{C}^n)$ is a fibre preserving map $\Phi: E \to E'$, such that the restriction of $\Phi$ to the fibre $F_x$ over $x \in X$ is a linear map for any $x \in X$. The corresponding map $\Phi_X: X \to X'$ is defined by $\pi_{E'} \circ \Phi = \Phi_X \circ \pi_E$. When $X = X'$ and $\Phi_X = \mathrm{id}_X$, it will be called an $X$-morphism. If $\Phi$ is invertible, it is said to be an isomorphism. In the particular case $E' = E$, an invertible morphism is an automorphism and if besides this, $\Phi_X = \mathrm{id}_X$, it is called an $X$-automorphism. The set of automorphisms of $E$ will be denoted $\mathrm{Aut}\, E$.

*Definition 1.* Let $E(X, \pi_E, \mathbb{C}^m)$ be a vector bundle and $G$ a Lie group. A vector bundle representation of $G$ on $E$ is a map $\alpha: G \to \mathrm{Aut}\, E$ such that

(i) $\alpha(g_1 \cdot g_2) = \alpha(g_1)\alpha(g_2)$,

(ii) the map $\tilde{\alpha}$; $G \times E \to E$, defined by $\tilde{\alpha}(g, u) = \alpha(g)u$, is differentiable.

A left action by automorphisms of a Lie group $G$ on a principal fibre bundle $P(X, H)$ is a differentiable action of $G$ on $P$ by the left such that $g(uh) = (gu)h$ for any $g \in G$, $u \in P$ and $h \in H$.

Given a linear representation $\sigma: H \to \mathrm{GL}(m, \mathbb{C})$ of $H$, for any left action by automorphisms of $G$ on the principal fibre bundle $P(X, H)$, there is a vector bundle representation of $G$ in the vector bundle $E(X, \pi_E, \mathbb{C}^m)$ associated to $P$ through $\sigma$, given by

$$\alpha(g)[u, v] = [g'u, v],$$

for any $g \in G$, $u \in P$ and $v \in \mathbb{C}^m$. The bracket $[u, v]$ means the equivalence class corresponding to $(u, v) \in P \times \mathbb{C}^m$ in the associate vector bundle $E$.

In the particular case of $H$ being a closed Lie subgroup of $G$, we can consider the principal fibre bundle $G(M, H)$ where $M$ is the homogeneous space $G/H$ and the left action of $G$ on itself by translations: the corresponding vector bundle representation of $G$ in the associated vector bundle $E_\sigma(M, \pi_\sigma, \mathbb{C}^m)$ is the vector bundle representation of $G$ induced by $\sigma$ (see e.g. Bott 1965).

To every vector bundle representation $\alpha$ of $G$ in $E(X, \pi_E, \mathbb{C}^m)$, we can associate a linear representation $\mathcal{U}_\alpha$ of $G$ in the linear space $\Gamma(E)$ of differentiable sections of $E$, as follows:

$$[\mathcal{U}_\alpha(g)\psi](gx) = \alpha(g)\psi(x). \tag{3.1}$$

*Definition 2.* A vector bundle representation of $(H, G)$ in $E(M, \pi_E, \mathbb{C}^m)$ is a vector bundle representation of $G$ in $E(M, \pi_E, \mathbb{C}^m)$ such that $\alpha_M$ is the usual action of $G$ on $M = G/H$.

The natural concept of equivalence between vector bundle representations of $(H, G)$ in $E(M, \pi_E, \mathbb{C}^m)$ is defined as follows.

*Definition 3.* Two vector bundle representations $\alpha$ and $\beta$ of $(H, G)$ in $E_\alpha$ and $E_\beta$ respectively are said to be equivalent if there is an $M$-isomorphism $\gamma$ of $E_\alpha$ in $E_\beta$ such that $\gamma \circ \alpha(g) = \beta(g) \circ \gamma$, $\forall g \in G$.

*Theorem 2.* If $\alpha$ and $\beta$ are equivalent vector bundle representations of $(H, G)$ in $E_\alpha$ and $E_\beta$ respectively, the associated linear representations $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ are equivalent.

*Proof.* Let $\gamma$ be the $M$-isomorphism connecting $\alpha$ and $\beta$. The the map $T: \Gamma(E_\alpha) \to \Gamma(E_\beta)$ given by $(T\psi)(x) = \gamma\psi(x)$ establishes the equivalence between $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$.

It is noteworthy that the induced vector bundle representations of $(H, G)$ exhaust, up to equivalence, all the vector bundle representations of $(H, G)$.

*Theorem 3.* Let $\alpha$ be a vector bundle representation of $(H, G)$ in $E$ and $x_0 = [e] = H$. The restriction of $\alpha(g)$ to $\pi_E^{-1}(x_0)$, $\sigma(g)$, defines a representation $\sigma: G_{x_0} \to \mathrm{Aut}\, \pi_E^{-1}(x_0)$ of $G_{x_0} = H$. There exists an equivalence between $\alpha$ and the vector bundle representation of $(H, G)$ induced by $\sigma$.

*Proof.* The fibre $\pi_E^{-1}(x_0)$ is invariant under $G_{x_0}$ and the restriction of $\sigma$ to $H$ is a morphism $\sigma: G_{x_0} \to \mathrm{Aut}\, \pi_E^{-1}(x_0)$. Now, let $E_\sigma(M, \pi_{E_\sigma}, \pi_E^{-1}(x_0))$ be the vector bundle associated to $G(M, H)$ through the representation $\sigma$; if $\rho: E_\sigma \to E$ is defined as $\rho[g, u] = \alpha(g)u$, $\forall g \in G$, $u \in \pi_E^{-1}(x)$, then $(\rho, \mathrm{id}_M)$ is an $M$-isomorphism intertwinning $\alpha$ with the natural action $\alpha^\sigma$ of $G$ on $E_\sigma$ because

$$\alpha(g')\rho[g, u] = \alpha(g'g)u = \rho[g'g, u] = \rho\alpha^\sigma(g')[g, u].$$

The inverse map $\rho^{-1}: E \to E_\sigma$ is defined by $\rho^{-1}(u) = [g, \alpha(g^{-1})u]$ for any $g$ such that $\pi_E(u) = gx_0$.

This theorem permits us to reduce the problem of equivalence between vector bundle representations to that of equivalence between induced representations.

*Theorem 4.* Two vector bundle representations $\alpha$ and $\beta$ of $(H, G)$ in $E_\alpha$ and $E_\beta$, respectively, are equivalent if and only if the representations $\sigma_\alpha$ and $\sigma_\beta$ of $H$ are equivalent.

*Proof.* If $\alpha$ and $\beta$ are equivalent, there exists an $M$-isomorphism $\rho$ of $E_\alpha$ in $E_\beta$ such that $\beta(g) \circ \rho = \rho \circ \alpha(g)$, $\forall g \in G$. Then $\rho$ maps $\pi_{E_\alpha}^{-1}(x_0)$ onto $\pi_{E_\beta}^{-1}(x_0)$ and the restriction of $\rho$ to $\pi_{E_\alpha}^{-1}(x_0)$ furnishes the equivalence between $\sigma_\alpha$ and $\sigma_\beta$.

Conversely, if $\sigma_\alpha$ and $\sigma_\beta$ are equivalent, there is an isomorphism $T$ of $\pi_{E_\alpha}^{-1}(x_0)$ in $\pi_{E_\beta}^{-1}(x_0)$ such that $T \circ \sigma_\alpha \circ T^{-1} = \sigma_\beta$. Then we define the isomorphism $\tau: E_{\sigma_\alpha} \to E_{\sigma_\beta}$ between the vector bundles associated to $G(M, H)$ through $\sigma_\alpha$ and $\sigma_\beta$, as follows: $\tau[g, v] = [g, Tv]$. It is very easy to check that if $\alpha_0$ and $\beta_0$ denote the actions of $G$ on $E_{\sigma_\alpha}$ and $E_{\sigma_\beta}$ respectively, then $\beta_0(g) \circ \tau = \tau \circ \alpha_0(g)$. Therefore $\tau$ is an equivalence between the vector bundle representations $\alpha_0$ and $\beta_0$ of $(H, G)$. The equivalence between the vector bundle representations $\alpha$ and $\beta$ follows from theorem 3 and transitivity of the equivalence relation.

Therefore, equivalence of vector bundle representations is reduced to a simpler problem: that of equivalence of finite-dimensional representations of $H$.

If the vector bundle $E(X, \pi_E, \mathbb{C}^m)$ admits a global trivialisation $\phi: X \times \mathbb{C}^m \to E$, we can write any section $\psi \in \Gamma(E)$ in terms of a function $F_\psi: X \to \mathbb{C}^m$, by means of

$$\phi(x, F_\psi(x)) = \psi(x). \tag{3.2}$$

Then if $\alpha$ is a vector bundle representation of $G$ in $E$, we can define a matrix function $A: G \times X \to \mathrm{GL}(m, \mathbb{C})$ as follows:

$$\alpha(g)\phi(x, a) = \phi(gx, A(g, x)a). \tag{3.3}$$

Property (i) of definition 1 implies that $A$ satisfies (2.2), and therefore the representation $\mathcal{V}_\alpha$ of $G$ defined by

$$\phi(gx, (\mathcal{V}_\alpha(g)F_\psi))(gx) = \alpha(g)\psi(x) \tag{3.4}$$

is a locally operating linear representation of $G$ with gauge matrix $A$. This construction is just the reciprocal of that given in theorem 1.

Let $\beta$ be any vector bundle representation of $G$ in $E$ equivalent to $\alpha$ and $S$ be the matrix function $S: M \to \mathrm{GL}(m, \mathbb{C})$ defined by

$$\gamma\phi(x, a) = \phi(x, S(x)a) \tag{3.5}$$

where $\gamma$ is the $M$-automorphism of $E$ such that $\alpha(g) \circ \gamma = \gamma \circ \beta(g)$, $\forall g \in G$. It is easy to check that $S$ establishes the gauge equivalence between $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$.

This shows that Hoogland's concepts of linear locally operating representations and their gauge equivalence correspond to that of vector bundle representations and their equivalence, respectively, provided $E$ admits a global trivialisation. Thus the theory of vector bundle representations and their equivalence permits us to generalise Hoogland's concepts by taking into account the topological features. Moreover, this intrinsic approach points out the reason for considering gauge equivalence: the representations (3.4) corresponding to the same vector bundle representation through two different trivialisations of $E$ are gauge equivalent.

## 4. Applications

Besides the standard applications to the theory of covariant and canonical representations of the Poincaré group (Asorey *et al* 1982), and to the theory of symmetry breakdown (Sen 1975, 1978), there are many other physically interesting vector bundle representations of symmetry groups. We shall discuss three of them in this section.

### *4.1. Wu–Yang monopole*

The Wu–Yang formulation of the Dirac magnetic monopole describes it by means of the Cartan–Killing connection $\omega$ of the principal fibre bundle $SU(2)(S^2, U(1))$ on the homogeneous space $SU(2)/U(1) \approx S^2$ (Wu and Yang 1975). We recall that $\omega$ is defined by the distribution of horizontal subspaces of the tangent bundle of SU(2) which are orthogonal to the U(1) fibres with respect to the Cartan–Killing metric of SU(2).

The quantum states of a charged particle moving in the magnetic field of a ($g = 1$) Dirac monopole are described by the cross sections of the linear bundle $E(S^1, \pi_E, \mathbb{C})$ associated to $SU(2)(S^2, U(1))$ by the natural representation of U(1) in $\mathbb{C}$. Now, in the bundle $SU(2)(S^2, U(1))$ there is the left action by automorphisms of SU(2) given by left translation. This action induces (see § 3) a vector bundle representation $\alpha$ of SU(2) in $E$ and a representation $\mathcal{U}_\alpha$ in the space of sections of $E$.

The dynamics of this system are governed by the Hamiltonian $H = -1/2\Delta_\omega$ (Asorey 1982). Now, since $\omega$ is invariant under the SU(2) left action, the representation $\mathcal{U}_\alpha$ of SU(2) leaves the Hamiltonian $H$ invariant, i.e. SU(2) is a dynamical symmetry of the quantum system.

In a similar way, it can also be shown that the system corresponding to magnetic monopoles with higher topological charge is defined in U(1) principal bundles which are inequivalent to $SU(2)(S^2, U(1))$. Hence, the corresponding vector bundle representations of SU(2) are not gauge equivalent to $\alpha$, which implies that they correspond to different physical systems (Asorey 1982). Indeed, the corresponding $\mathcal{U}_\alpha$ representations are inequivalent even as linear representations because by the Frobenius reciprocity theorem the irreducible SU(2) sectors contained in the corresponding $\mathcal{U}_\alpha$ representations are different.

In fact, $\mathcal{U}_\alpha$ is a representation of SU(2) induced from one of U(1). Next we consider two examples of vector bundle representations which are not induced from those of a proper subgroup.

### *4.2. Systems of two identical particles*

Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3; x = y\}$ be the diagonal subset of $\mathbb{R}^3 \times \mathbb{R}^3$. Any U(1) principal fibre bundle on $\mathbb{R}^3 \times \mathbb{R}^3 - \mathbb{D}$ is isomorphic to the trivial one $(\mathbb{R}^3 \times \mathbb{R}^3 - \mathbb{D}) \times U(1)$. In this principal bundle there are two different left actions $a_1, a_2$ of $\mathbb{Z}_2$ by automorphisms, given by

$$a_1(-1, (x, y, e^{i\theta})) = (y, x, e^{i\theta}) \qquad a_2(-1, (x, y, e^{i\theta}) = (y, x, e^{-i\theta})$$

respectively. They induce two vector bundle representations $\alpha_1, \alpha_2$ of $\mathbb{Z}_2$ in the vector bundle $E(\mathbb{R}^3 \times \mathbb{R}^3 - \mathbb{D}, \pi_E, \mathbb{C})$ associated to the trivial principal U(1) bundle $(\mathbb{R}^3 \times \mathbb{R} - \mathbb{D}) \times U(1)$ by the natural action of U(1) in $\mathbb{C}$. As in § 2, $\alpha_1$ and $\alpha_2$ define two representations $\mathcal{U}_{\alpha_1}, \mathcal{U}_{\alpha_2}$ of $\mathbb{Z}_2$ in the space $\Gamma(E)$ of sections of $E$.

The quantum states of the system formed by two indistinguishable particles are described by the sections of $\Gamma(E)$ which are invariant under $\mathcal{U}_{\alpha_1}$ or $\mathcal{U}_{\alpha_2}$ depending on whether the particles are bosons or fermions, respectively. The quantum dynamics behaviour of both kinds of systems is also invariant by $\mathcal{U}_{\alpha_1}$ or $\mathcal{U}_{\alpha_2}$, respectively. For this reason such representations are physically interesting.

### *4.3.* BPST *instanton*

The instanton discovered by Belavin, Polyakov, Schwarz and Tyupkin (1975) is a self-dual connection $\omega$ of a principal fibre bundle $P(S^4, \mathrm{SU}(2))$ with Pontrjagin index 1.

Let $E(S^4, \pi_E, \mathbb{C}^2)$ be the vector bundle associated to $P$ by the natural representation of SU(2) in $\mathbb{C}^2$. The cross sections of $E$ describe the quantum states of a particle moving under the action of the instanton Yang–Mills fields. Now, $P$ itself can be considered as the bundle associated to the spinor bundle Spin(5) $(S^4, \mathrm{Spin}(4))$ of $S^4$ by the homomorphism of groups

$$\Delta_+ : \mathrm{Spin}(4) \approx \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \to \mathrm{SU}(2)_+$$

and $\omega$ is the connection of $P$ induced by such a homomorphism from the spin connections of the Levi–Civita connection in $S^4$ (Atiyah *et al* 1978). Thus, the elements of $P$ can be considered as classes $[g, a]$ of pairs of elements of Spin(5) and SU(2). Then we can define the left action of Spin(5) in $P$ given by

$$g_1[g, a] = [g_1 g, a].$$

It is easy to show that this action is a left action of Spin(5) on $P$ by automorphisms. Therefore it induces a vector bundle representation $\alpha$ of Spin(5) on $E$ and a linear representation $\mathcal{U}_\alpha$ in the space of quantum states $\Gamma(E)$. This representation $\mathcal{U}_\alpha$ of Spin(5) points out the symmetry of the BPST system, which unifies the external symmetry SO(4) with the internal (gauge) one, SU(2). Finally, we remark that $\mathcal{U}_\alpha$ is not an induced representation of Spin(5).

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